DOUBLE SERIES

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Absract

We introduce double series and we shall give the definition of their convergence and divergence. Then we study the relationship between double and iterated series,
The theory of double series is intimately related to the theory of double sequences. To each double sequence z : N×N −→ C, there corresponds three important sums; namely:

1. $\sum_{n,m=1}^{\square} z(n,m)$,
2. $\sum_{n=1}^{\square}(\sum_{m=1}^{\square}(z(n,m))$
3. $\sum_{m=1}^{\square}(\sum_{n=1}^{\square}(z(n,m))$

And we give a sufficient condition for equality of iterated series.

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Introduction

Definition: Let z: N × N −→ C be a double sequence of complex numbers and let (s(n,m)) be the double sequence defined by the equation
s(n,m) :=$\sum_{i=1}^{n}(\sum_{j=1}^{m}(z(i,j))$
The pair (z,s) is called a double series and is denoted by the symbol$\sum_{n,m=1}^{\square} z(n,m)$ or, more briefly by$\sum z(n,m)$. Each number z(n,m) is called a term of the double series and each s(n,m) is called a partial sum. We say that the double series $\sum_{n,m=1}^{\square} z(n,m)$ is convergent to the sum s if $\lim_{n,m\to\square}$ s(n,m) = s. If no such limit exists, we say that the double series $\sum_{n,m=1}^{\square} z(n,m)$ is divergent.
The series $\sum_{n=1}^{\square}(\sum_{m=1}^{\square}(z(n,m))$ and $\sum_{m=1}^{\square}(\sum_{n=1}^{\square}(z(n,m))$ are called iterated series.
Theorem. If the double series $\sum_{n,m=1}^{\square} z(n,m)$ is convergent, then $\lim_{n,m\to\square}$ z(n,m) = 0.
Proof: Since the double series $\sum_{n,m=1}^{\square} z(n,m)$ is convergent, say to a, then its sequence of partial sums (s(n,m)) converges to a. So given ϵ > 0, there exists N ∈ **N** such that |s(n,m)−a| < ϵ/4 ∀n,m ≥ N. It follows that for all n,m ≥ N, we have
|z(n,m)| = |s(n,m) + s(n−1,m−1)−s(n,m−1)−s(n−1,m)| ≤ |s(n,m)−a|+|s(n−1,m−1)−a|+|s(n,m−1)−a| +|s(n−1,m)−a| <ϵ/ 4 + ϵ/ 4 + ϵ/ 4 + ϵ/ 4 = ϵ.
Therefore, limn,m→∞ z(n,m) = 0.
Theorem(Cauchy Convergence Criterion for Double Series.) A double series$\sum_{n,m=1}^{\square} z(n,m)$ of complex numbers converges if and only if its sequence of partial sums (s(n,m)) is Cauchy.
Proof: It follows immediately from the Definition and Cauchy Convergence Criterion for Double Sequences.
Double Series of Nonnegative Terms
Theorem. A double series of nonnegative terms $\sum_{n,m=1}^{\square} z(n,m)$ converges if and only if the set of partial sums {s(n,m) : n,m ∈ N} is bounded.
Proof: Let $\sum_{n,m=1}^{\square} z(n,m)$ be a double series with z(n,m) ≥ 0∀n,m ∈ N. If $\sum_{n,m=1}^{\square} z(n,m)$ converges, then its double sequence (s(n,m)) of partial sums converges and, hence, bounded, by Theorem (a convergent sequence of double sequence is bounded).It follows that the set{s(n,m) : n,m ∈ N} is bounded.
Suppose, conversely, that the set of partial sums {s(n,m) : n,m ∈ N} is bounded. Then the double sequence of partial sums (s(n,m)) is bounded. Since the terms z(n,m) of the double series are nonnegative, it is clear that the sequence (s(n,m)) is increasing. It follows from the Monotone Convergence Theorem that (s(n,m)) converges and, hence, $\sum_{n,m=1}^{\square} z(n,m)$ converges.
Corollary. A double series of nonnegative terms $\sum_{n,m=1}^{\square} z(n,m)$ either converges to a finite number s or else it diverges properly to ∞.
Proof: Let S := {s(n,m) : n,m ∈ N} be the set of partial sums of the double series $\sum_{n,m=1}^{\square} z(n,m)$. Then either the set S is bounded; that is, supS = s ≥ 0, and hence, by Monotone Convergence Theorem or by above Theorem, the sequence of partial sums (s(n,m)) converges to s and hence $\sum z(n,m)$ = s, or else the set S is unbounded, and in this case it is easy to show that $\lim_{n,m\to\square}$ s(n,m) = ∞, and hence the double series $\sum z(n,m)$ properly diverges to ∞.
Theorem(Comparison Test). Suppose that
0 ≤ u(n,m) ≤ v(n,m) for every n,m ∈ N.
(i). If $\sum_{n,m=1}^{\square} v(n,m)$ is convergent, then $\sum_{n,m=1}^{\square} u(n,m)$ is convergent.

(ii). If $\sum_{n,m=1}^{\square} u(n,m)$ is divergent, then $\sum_{n,m=1}^{\square} v(n,m)$ is divergent

Proof: (i). Suppose $\sum_{n,m=1}^{\square} v(n,m)$ is convergent, and let $\epsilon > 0$ be given. If (s0(n,m)) denotes the sequence of partial sums of the series $\sum_{n,m=1}^{\square} v(n,m)$, then there exists N ∈ **N** such that |s0(p,q)−s0(n,m)| < ϵ ∀ p ≥ n ≥ N and q ≥ m ≥ N,

and so if (s(n,m)) denotes the sequence of partial sums of the series $\sum_{n,m=1}^{\square} u(n,m)$, then we have |s(p,q)−s(n,m)|≤ (|s0(p,q)−s0(n,m)|) < ∀ p ≥ n ≥ N and q ≥ m ≥ N.

Hence, $\sum_{n,m=1}^{\square} u(n,m)$ converges.

(ii). Suppose $\sum_{n,m=1}^{\square} u(n,m)$ is divergent. Then, by Corollary , we have $\lim_{n,m\to\square}$ s(n,m) = ∞. Since, by hypothesis, s(n,m) ≤ s0(n,m) ∀ n,m ∈ N, it follows that $\lim_{n,m\to\square}$ s0(n,m) = ∞. Therefore, $\sum_{n,m=1}^{\square} v(n,m)$ diverges.

Definition. A double series $\sum_{n,m=1}^{\square} z(n,m)$P of complex numbers is said to be absolutely convergent if the double series $\sum_{n,m=1}^{\square} |z(n,m)|$ is convergent.

The iterated series $\sum_{n=1}^{\square}(\sum_{m=1}^{\square}(z(n,m))$ is said to be absolutely convergent if the series $\sum_{n=1}^{\square}(|\sum_{m=1}^{\square}(z(n,m)|)$is convergent.

Theorem. Every absolutely convergent double series is convergent.

Proof: Suppose that the double series $\sum_{n,m=1}^{\square} z(n,m)$ converges absolutely. Then $\sum_{n,m=1}^{\square} |z(n,m)|$ converges, and hence, by Cauchy Convergence Criterion , its sequence (s0(n,m)) of partial sums is Cauchy. So given $\epsilon > 0$, there exists N ∈ **N** such that |s0(p,q)−s0(n,m)| < ϵ ∀ p ≥ n ≥ N and q ≥ m ≥ N.

Letting (s(n,m)) denotes the sequence of partial sums of $\sum_{n,m=1}^{\square} z(n,m)$ it is easy to see that |s(p,q)−s(n,m)|≤|s0(p,q)−s0(n,m)| < ϵ ∀ p ≥ n ≥ N and q ≥ m ≥ N. It follows then that the sequence (s(n,m)) is Cauchy and therefore, by Cauchy Convergence Criterion , the series $\sum_{n,m=1}^{\square} z(n,m)$ converges

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